Jacobi's Formula and the Laplace-Beltrami operator

Ethan Y. Jaffe

The purpose of this note it to prove an identity from linear algebra called "Jacobi's Identity" and use it to give a description in coordinates of the Laplace-Beltrami operator. Recall that the Adjugate Matrix to a matrix g is defined by

$$g\,\mathrm{Adj}(g) = \det(g).$$

If g is invertible, this takes the pleasant form

$$\mathrm{Adj}(g) = g^{-1}\det(g).$$

Identity $M_n(\mathbf{C})$, the set of $n\times n$ matrices with $\mathbf{C}^{n^2}$. Then $\det : M_n(\mathbf{C}) \to \mathbf{C}$ is a differentiable map. Jacobi's formula lets one compute the derivative of this map. For simplicity of notation we state the identity for differentiable families.

Theorem 1 (Jacobi's Identity)**.** *Let $g(t) : \mathbf{R} \to M_n(\mathbf{C})$ be differentiable. Then*

$$\frac{d}{dt}\det(g(t)) = \mathrm{tr}\left(\mathrm{Adj}(g(t))\frac{d}{dt}g(t)\right).$$

Proof. To avoid difficult computations, we instead use a bit of analysis.

We first prove this for a very special family and a very speial t, $t = 0$ and $g(t) = tg + 1$. Then, either by direct computation or putting g into a normal form, we see that

$$\frac{d}{dt}\det(g(t))\bigg|_{t=0} = \mathrm{tr}\, g = \mathrm{tr}\left(\mathrm{Adj}(1)g\right).$$

Now we prove the formula under the assumption that $g(t)$ is invertible at $t = t_0$, which we assume without loss of generality to be $t_0 = 0$. Since det is differentiable and $g(t)$ is differentiable,

$$g(t) = g(0) + tg'(0) + o(t)$$

and so (for instance by the mean value theorem)

$$\det(g(t)) - \det(g(0) + tg'(0)) \in o(t).$$

Thus, from the definition of the derivative,

$$\frac{d}{dt}\det g(t)\bigg|_{t=0} = \lim_{t\to 0}\frac{\det g(t) - \det g(0)}{t}$$

$$= \lim_{t \to 0} \frac{\det(g(0) + tg'(0)) - \det(g(0))}{t} + o(1)$$
$$= \lim_{t \to 0} \frac{\det(g(0) + tg'(0)) - \det(g(0))}{t}.$$

We may rewrite

$$\det(g(0) + tg'(0)) = \det(g(0)) \det(1 + tg^{-1}(0)g'(0)),$$

and use the special case to deduce that

$$\frac{d}{dt} \det(g(0) + tg'(0))\Big|_{t=0} = \det(g(0)) \operatorname{tr}(g^{-1}(0)g'(0)) = \operatorname{tr}(\det(g(0))g^{-1}(0)g'(0)),$$

which is what we want since $\mathrm{Adj}(g(0) = \det(g(0))g^{-1}(0)$.

But what if $g(0)$ is not invertible? Consider instead the family for $s > 0$ defined by

$$g_s(t) = s + g(t).$$

One easily sees that $g_s(0)$ is invertible for s small. Either one can use a normal form, or notice that $\det(s + g(0))$ is a polynomial, so has isolated zeroes. So,

$$\frac{d}{dt} \det g_s(t)\Big|_{t=0} = \operatorname{tr}(\mathrm{Adj}(g_s(0))g'_s(0)).$$

Certainly $g_s(t), g'_s(t) \to g(t), g'(t)$, respectively, uniformly in t. Since $\operatorname{tr}, \mathrm{Adj}$, and multiplication are continuous, the right-hand side converges to

$$\operatorname{tr}(\mathrm{Adj}(g(0)g'(0)).$$

Since det is smooth, the chain rule implies that $(\det g_s)' \to (\det g)'$ uniformly (at least on compact sets in $M_n(\mathbf{C})$). In particular, the left-hand side converges to

$$\frac{d}{dt} \det g(t)\Big|_{t=0}.$$

This establishes the formula for $t = 0$, and hence for all t since the choice $t = 0$ was arbitrary. $\square$

We have the immediate useful Corollary.[1]

Corollary 2. *Let $U \subseteq \mathbf{R}^n$ be open, and let $g : U \to GL(n, \mathbf{C}) \subseteq M_n(\mathbf{C})$ be differentiable.[2]. Write the components of g as g_{ij}, and the components of its inverse as g^{ij}. Then*

$$g^{ij}\partial_k g_{ij} = \partial_k g_{ij} g^{ij} = \frac{\partial_k \det g}{\det g} = \partial_k \log(|\det g|).$$

[1]From here on, we will employ the Einstein summation convention, summing over repeated indices as long as one of the pair is "raised" and the other is "lowered."

[2]One may interpret this as either a map into a Lie group, or simply a differentiable map whose image consists only of invertible matrices

Proof. Fix $x \in U$ and consider the family $h(t) = g(x + te_k)$, where e_k is the kth basis vector. Then

$$h'(0) = \partial_k g(x)$$

and

$$(\det h)'(0) = \partial_k(\det g)(x).$$

By the Jacobi Formula,

$$(\det h)'(0) = \operatorname{tr}(\operatorname{Adj}(h(0))h'(0)) = \det(g(x)) \operatorname{tr}(g^{-1}(x)\partial_k g(x)) = \det(g(x))g^{ij}\partial_k g_{ij}.$$

The identity for the reversed order follows from the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, which we apply before the last equality. □

The application is as follows. Let M be a Riemannian (or pseudo-Riemannian) Manifold, with metric g and Levi-Civita connection D. Define the divergence of a vector field V by

$$\operatorname{div}(V) = \operatorname{tr}(Y \mapsto D_Y V)$$

and the Laplacian of a smooth function u by

$$\Delta u = \operatorname{div}(\operatorname{grad} u).$$

Choose local coordinates $\{x_1, \ldots, x_n\}$ for M. Write the components of a vector field V as $V = V^i \partial_i$, and the components of g by

$$g(\partial_i, \partial_j) = g_{ij}.$$

Write g^{ij} for the components of the inverse.

Proposition 3. *With the above definitions,*

$$\operatorname{div}(V) = \frac{1}{\sqrt{|\det g|}}\partial_i(\sqrt{|\det g|}V^i)$$

and

$$\Delta u = \frac{1}{\sqrt{|\det g|}}\partial_i(\sqrt{|\det g|}g^{ij}\partial_j u).$$

Proof. By definition $\operatorname{div}(V) = D_i V^i$, where $D_i V^i = (D_i V)^i$ denotes the ith component of $D_i V$ (as opposed to the connection applied to the smooth function V^i). Set Γ^k_{ij} to be the Christoffel symbols of the connection. Then

$$\operatorname{div}(V) = \partial_i V^i + \Gamma^i_{ij} V^j.$$

We know that

$$\Gamma^i_{ij} = \frac{1}{2}\left(g^{ik}\partial_i g_{ik} - g^{ik}\partial_k g_{ij} + g^{ik}\partial_j g_{ik}\right).$$

Since the sum runs over all i, k the symmetry of g and g^{-1} means that the first two terms cancel, and so by the Corollary

$$\Gamma^i_{ij} = \frac{1}{2} g^{ik} \partial_j g_{ik} = \partial_j \log \sqrt{|\det g|}.$$

Relabelling indices, we then have that

$$\operatorname{div}(V) = \partial_i V^i + \partial_i \log \sqrt{|\det g|} V^i = D_i V^i + \frac{\partial_i \sqrt{|\det g|}}{\sqrt{|\det g|}} V^i = \frac{1}{\sqrt{|\det g|}} \partial_i (\sqrt{|\det g|} V^i),$$

which is the desired formula.

In coordinates, it is clear that we have

$$\operatorname{grad} u = g^{ij} \partial_j u \partial_i,$$

since

$$g_{ij} g^{ki} \partial_k u V^j = g(\operatorname{grad} u, V) = du(V) = \partial_j u V^j.$$

Thus,

$$\Delta u = \frac{1}{\sqrt{|\det g|}} \partial_i (\sqrt{|\det g|} g^{ij} \partial_j u).$$

□